Computations for exercise 3

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Here we give some explicit computations for exercise 3, i.e. to show that the connecting homomorphism is well defined and gives a long exact sequence in cohomology for a short exact sequence of complexes.

One verification which we did not do yet was to show that the definition of $\partial_p([c])$ for $c \in Z^2(C^3_\bullet, d_\bullet)$ does not depend on the choice of a lifting $\tilde{c}$ of c to C^2_p. Indeed, let $\tilde{c}$ and $\tilde{c}'$ be two such liftings, then their difference $\tilde{c} - \tilde{c}'$ is a lifting of zero, thus there exists $b \in C^1_p$ such that $\varphi_p(b) = \tilde{c} - \tilde{c}'$. Now $d(\tilde{c} - \tilde{c}')$ defines the connecting homomorphism for the difference which is db, thus zero in cohomology.

Now we show the exactness of the long sequence in $H^p(C^2_\bullet, d_\bullet)$. As by exactness of the sequence of complexes $\psi \circ \varphi = 0$, we have $\mathrm{im}\,(\varphi) \subset \ker(\psi)$. Conversely, let $[c] \in H^p(C^2_\bullet, d_\bullet)$ be sent to zero under $\psi_* = H^p(\psi)$. Then there exists $c' \in C^3_p$ with $dc' = \psi_p(c)$. By surjectivity of ψ_{p-1}, there exists $c'' \in C^2_{p-1}$ with $\psi_{p-1}(c'') = c'$. Thus

$$\psi_p(c) = dc' = d\psi_{p-1}(c'') = \psi_p dc''.$$

Therefore $\psi_p(dc'' - c) = 0$, which implies by exactness that there exists $c''' \in C^1_p$ such that $\varphi_p(c''') = dc'' - c$. We conclude $[c] = [\varphi_p(c''')]$.

Now if the $c \in C^3_p$ in the definition of the connecting homomorphism is equal to $c = \psi_p(b)$ with $b \in Z^p(C_\bullet, d_\bullet)$, then we have already seen that this leads to $\partial_p[c] = 0$. Conversely, if $\partial_p[c] = 0$, then there exists $l \in C^1_p$ such that $\tilde{\tilde{c}} = dl$ (for the cocycle $\tilde{\tilde{c}}$ occurring in the definition of the connecting homomorphism). The intermediate cochain $d\tilde{c}$ becomes then $d\tilde{c} = d\varphi_p(l)$, and therefore $d(\tilde{c} - \varphi_p(l)) = 0$ and $\psi_p(\tilde{c} - \varphi_p(l)) = c$. Thus $\psi_*[\tilde{c} - \varphi_p(l)] = [c]$ and $[c]$ is in the image of ψ_*.

The last verification is that $\mathrm{im}\,(\partial_p) = \ker(\varphi_{p+1})$. If $\partial_p[c] = [a] \in H^{p+1}(C^1_\bullet, d_\bullet)$, then $\varphi_{p+1}(a) = d\tilde{c}$, thus $[\varphi_{p+1}(a)] = 0$. Conversely, if $\varphi_*[a] = 0$, then $\varphi_{p+1}(a) = d\tilde{c}$. Call $c := \psi(\tilde{c})$, then we have $\partial[c] = [a]$. This ends the proof.