Computations of the Cartan relations

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Let us compute the Cartan relations, i.e. for a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V, the following relations between the Lie derivative L_x with respect to an element $x \in \mathfrak{g}$, the insertion operator i_x and the Chevalley-Eilenberg differential d:

(a) $L_x = i_x \circ d + d \circ i_x$,

(b) $L_x \circ L_y - L_y \circ L_x = L_{[x,y]}$,

(c) $L_x \circ i_y - i_y \circ L_x = i_{[x,y]}$, and

(d) $L_x \circ d = d \circ L_x$.

Relation (a): We have on the one hand

$$\begin{aligned} d(i_x(c))(x_1,\ldots,x_p) &= \sum_{1\leq i<j\leq p} (-1)^{i+j} c(x,[x_i,x_j],\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_p) \\ &- \sum_{i=1}^{p} (-1)^i x_i \cdot c(x,x_1,\ldots,\widehat{x}_i,\ldots,x_p), \end{aligned}$$

while on the other hand

$$\begin{aligned} i_x(dc)(x_1,\ldots,x_p) &= \sum_{1\leq i<j\leq p} (-1)^{i+j} c([x_i,x_j],x,\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_p) \\ &- \sum_{i=1}^{p} (-1)^{i+1} x_i \cdot c(x,x_1,\ldots,\widehat{x}_i,\ldots,x_p) \\ &+ \sum_{j=1}^{p} (-1)^j c([x,x_j],\ldots,\widehat{x}_j,\ldots,x_p) + x\cdot c(x_1,\ldots,x_p). \end{aligned}$$

These last line gives $L_x c(x_1,\ldots,x_p)$, while the other terms cancel in the sum.

Relation (b): Computing $L_x(L_y c)(x_1,\ldots,x_p)$, we get terms with two brackets $[x,x_i]$ and $[y,x_j]$ as arguments of c, terms where x acts and $[y,x_i]$ is an argument, and terms where y acts and $[x,x_i]$ is an argument. These are symmetric in x and y with the same sign and cancel in the difference. The remaining terms are those where $[x,[y,x_i]]$ is an argument, those where $[y,[x,x_i]]$ is an argument and those where first x and then y acts and the contrary. By the Jacobi identity, these give rise to $L_{[x,y]}c(x_1,\ldots,x_p)$.

Relation (c): In $(L_x\circ i_y)(c)(x_1,\ldots,x_{p-1})$, L_x acts on the $(p-1)$-cochain i_yc. Thus there is no action of x on y. On the other hand, in $(i_y\circ L_x)(c)(x_1,\ldots,x_{p-1})$, L_x acts on the degree p cochain c which is evaluated on the list of elements $(y,x_1,\ldots,x_{p-1})$. We get exactly the same terms with the difference that there is a term with $[x,y]$ as the first argument. While all terms cancel, this last term gives $i_{[x,y]}c(x_1,\ldots,x_{p-1})$.

Relation (d): This is shown by induction on the degree of the cochain. For a degree 0 cochain, i.e. an element $v\in V$, we have on the one hand

$$L_y(dv)(x)=y\cdot dv(x)-dv([y,x])=y\cdot(x\cdot v)-[y,x]\cdot v,$$

and on the other hand

$$d(L_yv)(x)=d(y\cdot v)(x)=x\cdot(y\cdot v).$$

By the property that V is a $\mathfrak{g}$-module, the result of the first equation is equal to the result of the second.

For degree $p>0$, we can use insertion operators (which we cannot use in degree zero, because they diminish the degree !). We have

$$(d(L_xc)-L_xdc)(x_1,\ldots,x_{p+1})=i_{x_1}(d(L_xc)-L_xdc)(x_2,\ldots,x_{p+1}),$$

thus it suffices to show $i_x\circ d\circ L_y-i_x\circ L_y\circ d=0$. This is computed using the previous identities:

$$\begin{aligned}
i_x\circ d\circ L_y-i_x\circ L_y\circ d &= L_x\circ L_y-d\circ i_x\circ L_y+i_{[y,x]}\circ d-L_y\circ i_x\circ d\\
&= L_x\circ L_y-d\circ i_x\circ L_y+L_{[y,x]}-d\circ i_{[y,x]}-L_y\circ L_x+L_y\circ d\circ i_x\\
&= -d\circ i_x\circ L_y-d\circ i_{[y,x]}+L_y\circ d\circ i_x\\
&= -d\circ i_x\circ L_y-d\circ i_{[y,x]}+d\circ L_y\circ i_x\\
&= d\circ(i_x\circ L_y-i_{[y,x]}+L_y\circ i_x)=0.
\end{aligned}$$

Observe that we used the induction hypothesis from the third to the fourth line, using that the degree of the cochain we are operating on is one less, because i_x acts on it.