Exercice sheet 1: Homological and Lie algebra

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Exercice 1: Show that if there exists a contracting homotopy of the complex (C^*, d_*) , then its cohomology vanishes, i.e. $H^p(C^*, d_*) = 0$ for all $p \ge 0$.

Exercice 2: Show that a short exact sequence of complexes

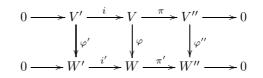
$$0 \to C_1^* \xrightarrow{\varphi^*} C_2^* \xrightarrow{\psi^*} C_3^* \to 0$$

induces a long exact sequence in cohomology, i.e. there exists a k-linear map $\partial^p: H^p(C_3^*) \to H^{p+1}(C_1^*)$, called the connecting homomorphism, such that the sequence of k-vector spaces

$$\dots \xrightarrow{\partial^{p-1}} H^p(C_1^*) \xrightarrow{H^p(\varphi)} H^p(C_2^*) \xrightarrow{H^p(\psi)} H^p(C_3^*) \xrightarrow{\partial^p} H^{p+1}(C_1^*) \to \dots$$

is exact.

Exercice 3: Suppose there is a commutative diagram of k-vector spaces



with exact rows and such that φ' and φ'' are isomorphisms. Show that φ is an isomorphism.

Exercice 4: Show that the kernel of a homomorphism of Lie algebras is an ideal. Construct a Lie algebra structure on the quotient vector space of a Lie algebra by an ideal. Have an example of a subalgebra of a Lie algebra which is not an ideal.

Exercice 5: Show that k is a g-module for any Lie algebra g. Show that g is a g-module for any Lie algebra g with respect to the "adjoint action", i.e. $x \cdot y := \operatorname{ad}_{x}(y) := [x, y]$ for all $x, y \in \mathfrak{g}$.

Exercice 6: Let \mathfrak{g} be a Lie algebra and V be a \mathfrak{g} -module. Show that the bracket defined by

$$[(a, x), (b, y)] := (x \cdot b - y \cdot a, [x, y]),$$

renders the vector space $V \oplus \mathfrak{g}$ a Lie algebra.