ICTP 2018 Summer School: Galois Theory Homework #1

1) For each of the following groups, write down all its subgroups: $\mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{Z}/6$, S_3, S_4. If you finished these, try the dihedral groups of order 8 and 10 and also S_5. Also, explicitly give all the elements of a Sylow 2-subgroup of A_4 as cycles. For instance one Sylow 3-subgroup is $\{e, (123), (132)\}$ and another is $\{e, (234), (243)\}$. Describe a Sylow 3-subgroup of S_6 explicitly.

2) Let $f(x) \in \mathbb{Z}[x]$ have degree d and roots $\alpha_1, \alpha_2, \ldots, \alpha_d$. The *discriminant* of $f(x)$ is $D := \left(\prod_{1 \le i < j \le d} (\alpha_i - \alpha_j) \right)^2$. So for $d = 3$, $D = ((\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3))^2$. What is the discriminant of $x^2 + bx + c$ in terms of b and c? What is the discriminant of $x^3 + bx + c$? (Recall that $b = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$ and $c = -\alpha_1\alpha_2\alpha_3$ where the α_i are the roots of $f(x)$. Note that the discriminant is invariant under the action of S_d, but that $\prod_{1 \le i < j \le d} (\alpha_i - \alpha_j)$ is *not* invariant under the action of S_d. Show it is invariant under the action of the alternating group $A_d \subset S_d$. Can you prove the fact below?

Fact Let $f(x) \in \mathbb{Z}[x]$ be irreducible of degree d. Let K be a splitting field of $f(x)$, let $G = Gal(K/\mathbb{Q})$ and let D be the discriminant of $f(x)$. Then $G \subseteq S_d$ and $G \subset A_d \iff D$ is a perfect square in $\mathbb{Z}$.

Show that $f(x) = x^3 + x^2 - 2x - 1$ is irreducible and has discriminant 7^2. Thus the splitting field K of $f(x)$ satisfies $Gla(K/\mathbb{Q}) \simeq \mathbb{Z}/3$. Can you find a quartic polynomial whose discriminant is a square?