ICTP 2018 Summer School: Galois Theory Homework #2

1) Let $\alpha_1, \alpha_2, \ldots, \alpha_5$ be the roots of the irreducible polynomial $x^5 - 2$. Show $\mathbb{Q}(\alpha_1)$ is not a splitting field of $f(x)$ but $\mathbb{Q}(\alpha_1, \alpha_2)$ is a splitting field of $f(x)$.

2) Factor $x^3 - 1$ to irreducible factors over $\mathbb{Q}$. Use Gauss' Lemma to make this easier. Do the same over $\mathbb{Z}/3$. Factor $x^5 - 1$ to irreducibles over $\mathbb{Z}/5$. What is the general result here?

3) In class I showed that the splitting field of $f(x) = x^4 - 10x^2 + 1$ is a degree 4 extension of $\mathbb{Q}$. Thus the Galois group of $f(x)$ is a *small* subgroup of S_4. What is the degree over $\mathbb{Q}$ of the splitting field of K of $f(x) = x^3 - 7$? Is the Galois group small or big? Let $L = \mathbb{Q}(\zeta_3)$. Show $K \subset L \subset \mathbb{Q}$ and $L/\mathbb{Q}$ and K/L are Galois.