ICTP 2018 Summer School: Galois Theory Homework #3

1) Let $K = \mathbb{Q}(\sqrt{-1})$. Write down all field automorphisms of K that fix $\mathbb{Q}$ pointwise, that is all bijective $\sigma : K \to K$ that satisfy:

- $\sigma(x + y) = \sigma(x) + \sigma(y)$,
- $\sigma(x \cdot y) = \sigma(x) \cdot \sigma(y)$,
- $x \in \mathbb{Q} \implies \sigma(x) = x$.

Do the same for $K = \mathbb{Q}(7^{1/4})$ and $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. One case is different from the other two. Why?

2) How many fields K are there satisfying $\mathbb{Q} \subset K \subset \mathbb{Q}(\zeta_p)$ when $p = 13$? How about when $p = 17$? For which type of primes p will there be few intermediate fields? For which will there be many?

3) In class I gave an example of an irreducible 5th degree polynomial that had three real roots and two complex roots. The Galois group of the splitting field over $\mathbb{Q}$ was then S_5 by a group theoretic lemma. Can you give an example of an irreducible 7th degree polynomial with exactly 5 real roots and two complex roots? If you can, then the Galois group of the splitting field over $\mathbb{Q}$ will be full group S_7.