ICTP 2018 Summer School: Galois Theory Homework #4

1) Let ζ_{11} be a primitive 11th root of unity. The minimal polynomial of ζ_{11} is $h(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. Let K be the splitting field of $h(x)$. Then $K/\mathbb{Q}$ is Galois with Galois group $(\mathbb{Z}/11)^* \simeq \mathbb{Z}/10$. Let $H \subset \mathbb{Z}/10$ be the unique subgroup of order 2. Then $K^H/\mathbb{Q}$ is a $\mathbb{Z}/5$-extension. Find it's minimal polynomial.

2) The extension $\mathbb{C}/\mathbb{R}$ is a $\mathbb{Z}/2$ Galois extension of fields. Complex conjugation generates the Galois group. Note that $N_{\mathbb{R}}^{\mathbb{C}}(i) = i \cdot -i = -i^2 = 1$ so Hilbert's Theorem 90 implies there exists a $z \in \mathbb{C}$ such that $i = z/\bar{z}$. Find such a z. Can you find them all?

3) In class we proved the fundamental theorem of algebra, that any $h(x) \in \mathbb{C}[x]$ factors to linear terms. This of course implies there are no finite extensions of $\mathbb{C}$. Alternatively, one can use Kummer theory to prove, for $n \geq 2$, there are no $\mathbb{Z}/n$-extensions of $\mathbb{C}$. Do this. Which result is stronger? (Remember, each nontrivial $\mathbb{Z}/n$-extension of $\mathbb{C}$ corresponds to a nontrivial element of $\mathbb{C}^*/(\mathbb{C}^*)^n$.