

# Advanced linear algebra

## (1) Jordan Normal form and Cayley Hamilton theorem

### Review of eigenvectors:

Let  $\mathbb{K}$  field,  $V$   $n$ -dimensional  $\mathbb{K}$ -vector space.

$A: V \rightarrow V$  endomorphism.

(equivalently  $A \in \text{Mat}(n, \mathbb{K}) = \{n \times n \text{ matrices in } \mathbb{K}\}$ )

Recall:  $\lambda \in \mathbb{K}$  is an eigenvalue of  $A$  if

$$\exists v \in V \setminus \{0\} \text{ with } A(v) = \lambda v.$$

$v$  is then called an eigenvector of  $f$  corresponding to  $\lambda$ .

The subspace  $\ker(A - \lambda I) \subset V$  is called the  $\lambda$ -eigenspace of  $V$ .

$\dim \ker(A - \lambda I)$  is called the geometric mult. of  $\lambda$ .

The characteristic polyn. of  $A$  is

$$p_A(x) = \det(xI - A), \quad (\text{polym. in } x \text{ of degree } \leq n).$$

The eigenvalues of  $A$  are the zeros of  $p_A(x)$ .

The algebraic multiplicity of  $\lambda$  is the multiplicity of  $\lambda$  as root of  $p_A(x)$ .

For a polynomial  $p(x) = a_n x^n + \dots + a_0$  we put  
 $p(A) := a_n A^n + \dots + a_0 I$ .

Cayley Hamilton theorem: Let  $R$  be a field,  $A \in \text{Mat}(n, R)$   
then  $p_A(A) = 0$ .

Proof: (Wrong proof:  $p_A(x) = \det(xI - A)$ , thus

$$p_A(A) = \det(AI - A) = \det(A - A) = \det(0) = 0.)$$

For the correct proof use adjugate matrix,

The adjugate matrix of a matrix  $A \in \text{Mat}(n, R)$  is  $\tilde{A} = (\tilde{A}_{ij})_{i,j=1}^n$

with  $\tilde{A}_{ij} = (-1)^{j+i} \det(A_{j,i})$  where  $A_{j,i}$  is the matrix

obtained from  $A$  by removing  $i^{\text{th}}$  row and  $j^{\text{th}}$  column.

Fact:  $A\tilde{A} = \tilde{A}A = \det(A)I$ . In particular if  $A$  is invertible

$$A^{-1} = \det(A)^{-1} \tilde{A}.$$

Set  $B = xI - A$ , then  $\det(B) = p_A(x)$

Set  $\tilde{B}$  be the adjugate matrix, then

$$\tilde{B}B = \det(B) \cdot I.$$

The entries of  $\tilde{B}$  are determinants of  $(n-1) \times (n-1)$  submatrices of  $B$  so they are polyn. in  $x$  of degree  $\leq n-1$

So we can write,

$$\tilde{B} = x^{n-1} B_{n-1} + x^{n-2} B_{n-2} + \dots + x B_1 + B_0$$

and have the identity

$$(x^{n-1} B_{n-1} + x^{n-2} B_{n-2} + B_0)(xI - A) = p_A(x) \cdot I = (x^n + b_{n-1}x^{n-1} + \dots + b_0)I$$

Comparing coeff. we get  $B_{n-1} = I$ ,  $B_{n-2} - B_{n-1}A = b_{n-1}I$ , ...

$$\dots B_0 - B_1A = b_1I, \quad -B_0A = b_0I.$$

Multiply these from the right by  $A^n, A^{n-1}, \dots$  and add.

$$\begin{array}{rcl} B_{n-1} A^n & & = A^n \\ B_{n-2} A^{n-1} - B_{n-1} A^n & & = b_{n-1} A^{n-1} \\ \vdots & & \\ B_0 A - B_1 A^2 & & = b_1 A \\ & - B_0 A & = b_0 I \end{array}$$

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$$0 = p_A(A).$$



Now assume  $\mathbb{K} = \mathbb{C}$

Main aim of this section:

Theorem (Jordan Normal form)  $V$   $\mathbb{K}$  vector space of dim  $n$ .

Let  $A: V \rightarrow V$  be an endomorph.

$\rightarrow \exists$  a basis  $v_1, \dots, v_n \in V$  s.t. wrt. this basis

$A$  is given by a block diag. matrix

$$J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{pmatrix}$$
 s.t. each  $J_i$  is a square matrix of the form.

$$J_i = \begin{pmatrix} \lambda & 1 & 0 \\ & \ddots & \vdots \\ 0 & & \lambda \end{pmatrix}$$
 where  $\lambda$  is an eigenvalue of  $A$ .

(Equivalently, let  $A \in \text{Mat}(n, \mathbb{C})$ , then  $A$  is similar to  $J$  as above (i.e.  $T^{-1}AT = J$  for some invertible  $n \times n$  matrix).

Example: Suppose  $A$  has  $k$ -distinct eigenvalues  $\lambda_j$  with geometric multiplicity  $g_j$  equal to the alg. multiplicity  $a_j$ .

Then each eigenspace  $\ker(A - \lambda_j I)$  has a basis of  $g_j$  eigenvectors.  $T$  has the union of

The basis has  $g_1 + \dots + g_n = a_1 + \dots + a_n = n = \dim V$  elements, thus is a basis of  $V$ .

In this basis  $A$  has the matrix

$$D = \begin{pmatrix} d_1 I_{g_1} & & \\ & \dots & \\ & & d_n I_{g_n} \end{pmatrix} \text{ with } I_j \text{ a } g_j \times g_j \text{ and matrix.}$$

We say  $A$  can be diagonalized.

Conversely if  $A$  can be diagonalized it has a basis of eigenvectors.

Example: Not all matrices can be diagonalized

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \text{ has only eigenvalue } 2, \text{ but} \\ \text{for } (A - 2I) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

## Decomposition into invariant subspaces

Definition: Set  $V_1, \dots, V_k$  subspaces of  $V$

We say  $V$  is the direct sum of  $V_1, \dots, V_k$ , denoted

$$V = V_1 \oplus \dots \oplus V_k, \text{ if every } v \in V \text{ can be written uniquely}$$

$$\text{as } v = v_1 + \dots + v_k \quad v_i \in V_i$$

Definition: A subspace  $W \subset V$  is called invariant under

$$A \in \text{End}(V) \Leftrightarrow AW \subset W.$$

Lemma Definition: Set  $M_A(x)$  be the unique monic polynomial of minimal degree with  $M_A(A) = 0$ .

$\forall p(x)$  is a polyn. with  $p(A) = 0$ , then  $M_A \mid p$ .

$M_A(x)$  is called the minimal polynomial of  $A$ .

In particular  $M_A(x)$  divides  $P_A(x)$  and all the zeros of  $M_A(x)$  are eigenvalues of  $A$ .

Proof. Division with  $r(x)$ .

$$p(x) = q(x)M_A(x) + r(x) \text{ with } r = 0 \text{ or } \deg r < \deg M_A(x)$$

By minimality  $r(x) = 0$ , thus  $M_A \mid p$ .

$\forall M_A, M'_A$  are two such polyn. then  $M_A \mid M'_A$  and  $M'_A \mid M_A$

as both are monic they are equal. //

Proposition: The zeros of  $M_A(x)$  are precisely the eigenvalues of  $A$ .

Proof: By above  $M_A(x) \mid P_A(x)$ , so all the zeros of  $M_A(x)$  are eigenvalues

Conversely let  $\lambda$  be an eigenvalue of  $A$  and  $v$  a corresp. eigenvector. Writing  $M_A(x) = a_m x^m + \dots + a_0$  we get

$$0 = M_A(A)v = \sum_{j=0}^m a_j A^j v = \sum_{j=0}^m a_j \lambda^j v = M_A(\lambda)v. //$$

By the fundamental thm of algebra

$M_A(x) = (x - \lambda_1)^{m_1} \cdot \dots \cdot (x - \lambda_k)^{m_k}$  with  $\lambda_1, \dots, \lambda_k$  the eigenvalues of  $A$ .

Recall: Two polynomials  $p_1(x), p_2(x)$  are relatively prime

if there is no polynomial of positive degree  $p$  with

$p | p_1$  and  $p | p_2$

(equivalent over  $\mathbb{C}$ ,  $p_1$  and  $p_2$  have no common zeros)

Fact: If  $p_1(x), p_2(x)$  are relatively prime, then there

exist polynomials  $b_1(x), b_2(x)$  with  $p_1(x)b_1(x) + p_2(x)b_2(x) = 1$ .

Lemma: Suppose  $p(x) = p_1(x)p_2(x)$  with  $p_1, p_2$  relatively prime.

If  $p(A) = 0$ , we have

$V = \ker p_1(A) \oplus \ker p_2(A)$  and  $\ker p_1(A)$  and  $\ker p_2(A)$

are both invariant under  $A$ .  $\rightarrow \checkmark \rightarrow$

Proof:

Invariance: If  $v \in \ker p_j(A)$ , then

$$p_j(A)Av = Ap_j(A)v = 0, \text{ then } Av \in \ker p_j(A).$$

As  $p_1(x), p_2(x)$  are rel. prime.  $\exists$  polyn.  $q_1(x), q_2(x)$

$$\text{with } p_1(x)q_1(x) + p_2(x)q_2(x) = 1.$$

For  $v \in V$  put  $v_1 = p_2(A)q_2(A)v$ ,  $v_2 = p_1(A)q_1(A)v$ .

Then  $v = v_1 + v_2$  and

$$p_2(A)v_2 = p_2(A)p_1(A)q_1(A)v = q_1(A)p_1(A)v = 0$$

Thus  $v_2 \in \ker p_2(A)$ , similarly  $v_1 \in \ker p_1(A)$ .

So  $V = \ker p_1(A) + \ker p_2(A)$ .

Finally if  $x_1 + x_2 = x_1' + x_2'$  with  $x_i, x_i' \in \ker p_i(A)$

Then  $u = x_1 - x_1' = x_2 - x_2' \in \ker p_1(A) \cap \ker p_2(A)$

Therefore  $u = q_1(A)p_2(A)u + q_2(A)p_1(A)u = 0$ .

Therefore  $V = \ker p_1(A) \oplus \ker p_2(A)$

Theorem: Assume  $M_A(x) = (x-d_1)^{m_1} \cdots (x-d_r)^{m_r}$   $d_i$  distinct

Then  $V = \ker (A-d_1I)^{m_1} \oplus \cdots \oplus \ker (A-d_rI)^{m_r}$

Furthermore each space  $\ker (A-d_jI)^{m_j}$  is invariant under  $A$ .

Proof: We make induction on  $k$ . The case  $k=1$  is trivial

$$\text{Set } g(x) := (x - \lambda_2)^{m_2} \cdots (x - \lambda_k)^{m_k}.$$

Then  $(x - \lambda_1)^{m_1}$  and  $g(x)$  are relatively prime.

Applying the lemma to  $(x - \lambda_1)^{m_1}$  and  $g(x)$  we find that

$$V = \ker(\lambda_1 I - A)^{m_1} \oplus U \quad \text{with } U = \ker g(A), \text{ and both are } \text{A-invariant}$$

In particular  $g(A)|_U = 0$ . Thus by

$$\text{induction } U = \ker(\lambda_2 I - A)^{m_2} \oplus \cdots \oplus \ker(\lambda_k I - A)^{m_k}.$$

Definition: The subspace

$\ker(A - \lambda_j I)^{m_j}$  is called the generalized eigenspace for  $\lambda_j$ . A vector  $u \in \ker(A - \lambda_j I)^{m_j}$  is called a generalized eigenvector.

Lemma:  $m_j$  is the smallest  $m > 0$  with

$$(A - \lambda_j I)^m \Big|_{\ker(A - \lambda_j I)^{m_j}} = 0.$$

Proof: Clearly  $(A - \lambda_j I)^{m_j} \Big|_K = 0$ .

Conversely assume  $(A - \lambda_j I)^{m_j-1} u = 0 \quad \forall u \in \ker(A - \lambda_j I)^{m_j}$

Write  $v \in V$  as  $v = v_1 + \tilde{v}$  according to the decomp.

$$V = \ker(A - \lambda_j I)^{m_j} \oplus \ker \tilde{p}(A) \text{ with}$$

$$\tilde{p}(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_j)^{m_j} \cdots (x - \lambda_s)^{m_s}.$$

$$\text{Then } (A - \lambda_j I)^{m_j-1} \tilde{p}(A) v = \tilde{p}(A) (A - \lambda_j I)^{m_j-1} v_1 + (A - \lambda_j I)^{m_j-1} \tilde{p}(A) \tilde{x} = 0.$$

Contradicting the definition of the minimal polynomial.  $\square$

## End of the proof: Nilpotent endomorphisms

Selecting a basis  $\{u_{j,1}, \dots, u_{j,m_j}\}$  for each gen. eigenspace  $\mathcal{E}_\lambda (A - \lambda_j I)^{m_j}$ , their union is a basis of  $V$ .

As each gen. eigenspace is invariant under  $A$ , in this basis  $A$  is block diagonal of the form.

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{pmatrix} \text{ with each } A_j \text{ an } n_j \times n_j \text{ matrix.}$$

To show Jordan Normal form we have to show we can choose the basis  $u_{j,1}, \dots, u_{j,m_j}$  in such a way that  $A_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_j \end{pmatrix}$  with  $\lambda_j$  of the form  $\begin{pmatrix} \lambda_j & 1 & 0 \\ & \lambda_j & \vdots \\ 0 & & \lambda_j \end{pmatrix}$

Definition: An endomorphism  $N: V \rightarrow V$  is called nilpotent if there exists an  $m > 0$  with  $N^m = 0$ .

Returning  $A$  to the generalized eigenspace  $\mathcal{E}_\lambda (A - \lambda_j I)^{m_j}$  can assume  $A$  has only one eigenvalue  $\lambda := \lambda_j$ .

We put  $N := A - \lambda_j I$ ,  $m = m_j$ ,  $V = \mathcal{E}_\lambda (A - \lambda_j I)^m$ ,  $n = \dim V$ .

Then  $N$  is nilpotent, in fact we have seen that  $m$  is minimal with  $N^m = 0$ . Thus  $M_A(z) = (z - \lambda)^m$ .

Definition: Let  $l > 0$ , A set of nonzero vectors.

$u, Nu, \dots, N^{l-1}u$  with  $N^l u = 0$  is called a Jordan chain.

Claim. There is a basis of  $V$  which is a union of Jordan chains  $\{u_i, Nu_i, \dots, N^{l_i-1}u_i\}$   $i = 1, \dots, s$

The claim implies the Theorem:

In this basis is of the form  $N = \begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_s \end{pmatrix}$  with  $N_i$  the  $l_i \times l_i$  matrix  $\begin{pmatrix} 0 & 1 & 0 \\ & \ddots & \vdots \\ 0 & & -1 \\ & & & 0 \end{pmatrix}$  and  $A$  is of the form  $(N + dI)$

Proof of the claim:  $n = \dim V$ , clearly  $n \geq m$ .

If  $n = 1$ , the claim is trivial.

Assume  $n \geq 2$ , and claim holds for all  $\mathbb{C}$  vector spaces of dim  $n$ . As  $N$  is nilpotent, it is not invertible.

Thus  $\dim(\text{im } N) < n$ .

By induction hypothesis there is a basis of Jordan chains  $u_i, Nu_i, \dots, N^{l_i-1}u_i$   $i = 1, \dots, s$  for  $\text{im}(N)$ .

For each  $u_i$  choose  $v_i \in V$  with  $N(v_i) = u_i$

Thus we have extended each Jordan chain by 1 element.

Claim:  $v_i, Nv_i, \dots, N^{l_i}v_i$   $i=1, \dots, t$  are linearly independent.

Proof of claim  
 suppose  $\sum_{i=1}^t \sum_{j=0}^{l_i} \alpha_{ij} N^j v_i = 0$

Applying  $N$  gives  $\sum_{i=1}^t \sum_{j=0}^{l_i-1} \alpha_{ij} N^{j+1} v_i = 0$

By induction hypothesis we have  $\alpha_{ij} = 0$  for  $i=1, \dots, t$   
 $0 \leq j \leq l_i - 1$ .

This gives  $\sum_{i=1}^t \alpha_{i, l_i} N^{l_i} v_i = \sum_{i=1}^t \alpha_{i, l_i} N^{l_i-1} u_i = 0$

By induction hypothesis again  $\alpha_{i, l_i} = 0$   $1 \leq i \leq t$ .

Finally extend the vectors of (\*) to a basis of  $V$  by adding vectors  $\tilde{w}_1, \dots, \tilde{w}_k$

For each  $i=1, \dots, k$ , we have  $N\tilde{w}_i \in \text{range}(N)$

Thus there exists a vector  $\bar{w}_i \in \langle v_i, \dots, N^{l_i}v_i \mid i=1, \dots, t \rangle$

with  $N\tilde{w}_i = N\bar{w}_i$ .

Set  $w_i = \tilde{w}_i - \bar{w}_i$ . Then

$v_i, \dots, N^{l_i}v_i, i=1, \dots, t, w_1, \dots, w_k$  form a basis of

$V$  consisting of Jordan chains.

(Note  $Nw_i = N\tilde{w}_i - N\bar{w}_i = 0$ , thus  $w_i$  are Jordan chains of length 1.)

Remark: The Jordan normal form  $J$  of  $A$  is not unique, but the number of blocks and their sizes are determined by  $A$ .

Proof: It suffices to consider a nilpotent operator  $N: V \rightarrow V$ . Set  $\beta(i) = \# \{k \times k \text{ blocks in } JNF\}$

Exercise:  $\beta(i) = \dim(\ker N |_{\text{im } N^{i-1}}) - \dim(\ker N |_{\text{im } N^i})$ .

### Matrix exponential

Let  $A \in \text{Mat}(n, \mathbb{C})$ . Consider the initial value problem

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0 \quad \text{for } x: \mathbb{R} \rightarrow \mathbb{C}^n$$

Solution is  $x(t) = e^{tA} \cdot x_0$

If  $J$  is in JNF and  $A = T J T^{-1}$   $T$  invertible

$$\text{Then } e^{tA} = T e^{tJ} T^{-1}$$

and if  $J_1, \dots, J_s$  are the Jordan blocks of  $J$ , then

$$e^{tJ} = \begin{pmatrix} e^{tJ_1} & & \\ & \ddots & \\ & & e^{tJ_s} \end{pmatrix}$$

Exercise: If  $y_j = \begin{pmatrix} \lambda_j^1 & 0 \\ \vdots & \vdots \\ 0 & \lambda_j^k \end{pmatrix}$ , then

$$e^{ty_j} = e^{\lambda_j t} \cdot \begin{pmatrix} 1 & t & t^2/2 & \dots & t^{k-1}/(k-1)! \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & \vdots & 1 \end{pmatrix}$$

### Jordan Normal form over $\mathbb{R}$

Let  $V$  be a f.d.  $\mathbb{R}$ -vector space,  $f: V \rightarrow V$  endom.

There is a basis of  $V$  s.t. wr.t. this basis the matrix of  $f$  is in block diagonal form.

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_r \end{pmatrix}$$

With each  $A_i$  of the form  $\begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$  for an eigenvalue  $\lambda$  of  $A$

or of the form.

$$\begin{pmatrix} \lambda - \mu & 1 & 0 & \dots & 0 \\ \mu & \lambda & 0 & 1 & \dots & 0 \\ 0 & 0 & \lambda & -\mu & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \mu & \lambda & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & \dots & \dots & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \lambda - \mu & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \lambda - \mu \end{pmatrix}$$

Sketch of the proof: Over  $\mathbb{C}$   $P_f(x)$  splits into

linear factors. Some of them are of the form  $x - d$  with  $d \in \mathbb{R}$ .

The others are of the form  $x - (d + \mu i)$   $d, \mu \in \mathbb{R}, \mu \neq 0$ .

The latter ones occur in pairs

$$(x - (d + \mu i))(x - (d - \mu i)) = x^2 + 2dx + d^2 + \mu^2$$

If  $v_1, \dots, v_m$  is a basis for the gen. eigenspace (over  $\mathbb{C}$ ) for the eigenvalue  $d + \mu i$ , then  $\bar{v}_1, \dots, \bar{v}_m$  is a basis

for the gen. eigenspace corresp. to  $d - \mu i$ .  $\bar{v}$  complex conj.

The vectors  $\frac{1}{2}(v_1 + \bar{v}_1), \frac{1}{2i}(v_1 - \bar{v}_1), \dots, \frac{1}{2}(v_m + \bar{v}_m), \frac{1}{2i}(v_m - \bar{v}_m)$

There are in  $\mathbb{R}^n$  and form a basis of the ~~sum~~ of the

two gen. eigenspaces for  $d + \mu i, d - \mu i$ .

And in this basis  $f$  is given by this generalised

Jordan Block. 

## Symmetric bilinear forms

Definition  $V$  finite-dim  $\mathcal{K}$ -vector space

A symmetric bilinear form on  $V$  is

$\phi : V \times V \rightarrow \mathcal{K}$  which is

(1) linear in both arguments e.g.  $\phi(\alpha x + \alpha' x', y) = \alpha \phi(x, y) + \alpha' \phi(x', y)$

(2)  $\phi(x, y) = \phi(y, x)$

Example: Standard inner product  $(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = \sum_{i=1}^n x_i y_i$   
on  $\mathbb{R}^n$ .

Definition: The kernel of a SBLF  $\phi$  on  $V$  is

$$\ker \phi := \{w \in V \mid \phi(v, w) = 0 \ \forall v \in V\}$$

$\phi$  is called nondegenerate if  $\ker \phi = \{0\}$ .

Definition (Matrix of a SBLF.) Set  $v_1, \dots, v_n$  basis of  $V$

$\phi$  SBLF on  $V$ . The matrix of  $\phi$  wrt  $v_1, \dots, v_n$  is

$$A = (a_{ij})_{i,j=1}^n \quad \text{with} \quad a_{ij} = \phi(v_i, v_j)$$

If  $V = \mathcal{K}^n$  the matrix of  $\phi$  is  $A = (\phi(e_i, e_j))_{i,j}$

If  $x = \sum_{i=1}^n \alpha_i v_i$ ,  $y = \sum_{j=1}^n \beta_j v_j$ , then

$$\phi(x, y) = (\alpha_1 \dots \alpha_n) A \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\phi(x, y) = \sum_{i,j} \alpha_i \beta_j a_{ij}$$

Remark: (1)  $A$  is symmetric:  $A = A^t$

(2)  $\phi$  is nondegenerate  $\Leftrightarrow \text{rank } A = \dim V \Leftrightarrow \det(A) \neq 0$

(1) is obvious, (2) this is because  $x = \sum \alpha_i v_i \in \ker \phi$   
 $\Leftrightarrow \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \ker A$ .

Example: For the standard inner product on  $\mathbb{R}^n$  the corresp. matrix is  $I_n$ .

Recall: dual vector space

For a vector space  $V$ , the dual vector space is

$$V^* = \{ f: V \rightarrow \mathbb{R} \mid f \text{ linear} \}$$

If  $V$  has dimension  $n$  and  $v_1, \dots, v_n$  is a basis, then

$v_1^*, \dots, v_n^*$  is the dual basis of  $V^*$  with

$$v_i^* \left( \sum_{j=1}^n \alpha_j v_j \right) = \alpha_i$$

Lemma: Let  $V$  be a finite dim  $\mathbb{R}$ -vector space, and  $\phi$  a nondeg. bilinear form on  $V$ .

$\Rightarrow$  Every linear form  $\psi \in V^*$  can be represented as  $\psi(\cdot) = \phi(v, \cdot): W \mapsto \phi(v, w)$  for a unique  $v \in V$ .

Proof: The map  $\tilde{\phi}: V \rightarrow V^*; v \mapsto \phi(v, \cdot)$  is linear and its kernel is  $\ker \phi = \{0\}$ .

Thus it is an isomorphism.

Proposition/Definition: Let  $\phi$  be a symm Bf. on vector space  $V$  of dim  $n$ .

Let  $U \subset V$  be a subspace, s.t.  $\phi|_{U \times U}$  is nondeg.

The orthogonal complement of  $U$  is

$$U^\perp = \{u \in V \mid \phi(u, v) = 0 \ \forall u \in U\}$$

If  $u_1, \dots, u_p$  is a basis of  $U$  and  $v_1, \dots, v_s$  a basis of  $U^\perp$  then w.r.t. to  $u_1, \dots, u_p, v_1, \dots, v_s$  the matrix of  $\phi$

$\phi$  is block diagonal.  $\begin{pmatrix} A_U & 0 \\ 0 & A_{U^\perp} \end{pmatrix}$

Proof: <sup>(1)</sup>  $U \cap U^\perp = \{0\}$  because if  $v \in U \cap U^\perp$  then  $\phi(u, v) = 0 \ \forall u \in U \xRightarrow{\phi \text{ nondeg}} v = 0$ .

(2)  $U + U^\perp = V$ : Set  $v \in V$ , then by proposition

there exists  $u' \in U$  with  $\phi(v, u) = \phi(u', u) \ \forall u \in U$ .

Thus  $\phi(v-u', u) = 0 \quad \forall u \in U$ . Therefore  $v-u' \in U^\perp$ .

Thus  $v = u' + (v-u') \in U + U^\perp$ .

## Classification of symmetric Blf

Theorem: Let  $\mathbb{K} = \mathbb{R} \neq \mathbb{C}$ . Let  $V$   $n$ -dim  $\mathbb{K}$ -vector space  
 $\phi$  symm. Blf on  $V$ .

There is a basis  $v_1, \dots, v_n$  of  $V$  s.t. w.r.t. to it.

$\phi$  is represented by a diagonal matrix.

Equivalently: Every symmetric matrix  $A \in \text{Mat}(n, \mathbb{K})$   
is congruent to a diagonal matrix.

(i.e.  $\exists$  invertible matrix  $P$  s.t.  $P^T A P = D$  diagonal)

Proof: If  $\phi = 0$  the claim is trivial. So let  $\phi \neq 0$ .

Claim: There exists a  $v \in V$  with  $\phi(v, v) \neq 0$

As  $\phi \neq 0$  there exists  $v, w \in V$  with  $\phi(v, w) \neq 0$ .

Note  $0 \neq 2\phi(v, w) = \phi(v+w, v+w) - \phi(v, v) - \phi(w, w)$

thus one of  $\phi(w, v)$ ,  $\phi(w, w)$ ,  $\phi(v+w, v+w)$  must be nonzero.

Now induction on  $n = \dim V$ .

Choose  $v_1 \in V$  with  $\phi(v_1, v_1) \neq 0$ . Set  $U = \langle v_1 \rangle$ .

Then  $\phi$  is nondegenerate on  $U$ . Thus  $V = \langle v_1 \rangle \oplus U^\perp$ .

By induction  $U^\perp$  has a basis, such that  $\phi|_{U^\perp \times U^\perp}$  is repr. by diag. matrix  $D_1$ .

Thus  $\phi$  is repr. by  $\begin{pmatrix} \phi(v_1, v_1) & 0 \\ 0 & 0 \end{pmatrix}$ .

For  $\mathbb{K} = \mathbb{C}, \mathbb{R}$  we can get a very simple description.

Theorem: (Classification of symm. Bf. over  $\mathbb{C}$ ).

Let  $V$   $n$ -dim  $\mathbb{C}$  vector space with symm. Bf.  $\phi$ .

Then there exists a basis with which  $\phi$  is given by

$\left( \begin{array}{c|c} \mathbb{I}_r & 0 \\ \hline 0 & 0 \end{array} \right)$  and the rank  $0 \leq r \leq n$  is uniquely determined by  $\phi$ .

Proof By previous theorem there is a basis with which  $\phi$  is

given by diagonal matrix  $D$ , we can put the zeros last.

Set  $d_{ii}$  be a nonzero diagonal entry. Multiply corresp.

basis vector by  $\frac{1}{\sqrt{d_{ii}}}$ . Note  $r$  is just the rank of  $D$ .



## Inner product spaces

Inner products serve to study lengths and angles in vector spaces. They play a big role in many fields from differential geometry to functional analysis.

Definition: Let  $V$  be an  $\mathbb{R}$ -vector space. An inner product on  $V$  is a pos. definite sym  $B$  of on  $V$ . Usually write it as  $\langle x, y \rangle$ .

An  $\mathbb{R}$ -vector space with inner product is called real inner product space.

Also consider the case of  $\mathbb{C}$ -vector spaces, still want positive definiteness  $\langle x, y \rangle \in \mathbb{R}_{>0}$  for all  $x \in V \setminus \{0\}$

Thus  $\langle, \rangle$  cannot be linear because then would have

$$\langle i x, i x \rangle = - \langle x, x \rangle.$$

So we use Hermitian forms.

Definition: Let  $V$  be a  $\mathbb{C}$ -vector space.

A Hermitian form on  $V$  is a map  $\langle, \rangle: V \times V \rightarrow \mathbb{C}$

(1) linear in first argument, conjugate linear in second:

$$\langle \lambda x + \lambda' x', y \rangle = \lambda \langle x, y \rangle + \lambda' \langle x', y \rangle$$

$$\langle x, \lambda y + \lambda' y' \rangle = \bar{\lambda} \langle x, y \rangle + \bar{\lambda}' \langle x, y' \rangle.$$

complex conjugates

(2) Conjugate symmetric

$$\langle y, x \rangle = \overline{\langle x, y \rangle}$$

(3) Positive definite

$$\langle x, x \rangle \in \mathbb{R}_{>0} \quad \forall x \in V \setminus \{0\}$$

A  $\mathbb{C}$ -vector space with a Hermitian form is called a complex inner product space. A real or complex inner product space is just called an inner product space.

Definition: (Norm associated to inner product)

Let  $V$  inner product space.

(1) For  $x \in V$  put  $\|x\| = \sqrt{\langle x, x \rangle} \in \mathbb{R}_{>0}$ .

$x$  is called a unit vector if  $\|x\| = 1$ .

(2)  $x, y \in V$  are called orthogonal, denoted  $x \perp y$  if  $\langle x, y \rangle = 0$ .

(3) A subset  $S \subset V$  is called orthogonal if  $x \perp y$  for  $x \neq y \in S$ .  
 $S$  is called orthonormal if in addition  $\|x\| = 1 \quad \forall x \in S$ .

(4)  $v_1, \dots, v_n \in V$  is an orthonormal basis of  $V$  if  $\{v_1, \dots, v_n\}$  is orthonormal and a basis.

The norm  $\|\cdot\|$  has the usual properties you know from analysis

Proposition: Set  $V$  inner product space.

(1) For  $x \in V$ ,  $\lambda$  scalar we have  $\|\lambda x\| = |\lambda| \|x\|$ .

(2) (Cauchy-Schwarz inequality):

For  $x, y \in V$  have  $\langle x, y \rangle \leq \|x\| \|y\|$

with equality  $\Leftrightarrow$   $x, y$  are linearly dependent.

(3) (Triangle inequality). For  $x, y \in V$  we have

$$\|x+y\| \leq \|x\| + \|y\|$$

Proof: Exercise.

Now show: Every finite-dim inner product space has orthonormal basis. In fact can modify any basis to make it orthonormal.

Theorem (Gram-Schmidt orthonormalization).

Set  $V$  finite dim inner product space,  $x_1, \dots, x_n$  basis of  $V$ .

For  $0 \neq x \in V$  write  $x' = \frac{x}{\|x\|}$ , s.t.  $\|x'\| = 1$ .

Define  $y_1 = x'_1$ ,  $y_2 = (x'_2 - \langle x'_2, y_1 \rangle y_1)$

inductively  $y_{i+1} = (x'_{i+1} - \sum_{j=1}^i \langle x'_{i+1}, y_j \rangle y_j)$

Then  $y_1, \dots, y_n$  is an ONB of  $V$ .

In particular every finite-dim inner product space has an ONB.

Proof: Induction on  $n$

Case  $n=1$ :  $x_1 \neq 0$ , then  $y_1 = \frac{x_1}{\|x_1\|}$  is ONB.

Set  $n \geq 2$ . By induction  $y_1, \dots, y_{n-1}$  is an ONB for  $\langle x_1, \dots, x_{n-1} \rangle$ .

$$\text{Set } z := x_n - \sum_{i=1}^{n-1} \langle x_n, y_i \rangle y_i$$

As  $x_1, \dots, x_n$  are linearly indep. we have  $z \neq 0$

Furthermore for  $0 \leq j \leq n-1$  have

$$\langle z, y_j \rangle = \langle x_n, y_j \rangle - \sum_{i=1}^{n-1} \langle x_n, y_i \rangle \langle y_j, y_i \rangle,$$

$$= \langle x_n, y_j \rangle = \langle x_n, y_j \rangle \text{ because } y_1, \dots, y_{n-1} \text{ are orthog.}$$

Now put  $y_n = \frac{z}{\|z\|}$ , then  $\langle y_n, y_j \rangle = 0$  for  $j < n$  and  $\|y_n\| = 1$ .

Corollary: Let  $V$  be an  $n$ -dim inner product space and  $v_1, \dots, v_k$  an orthonormal set. Then it can be extended to an orthonormal basis  $v_1, \dots, v_n$  of  $V$ .

## Self adjoint operators

Definition: Let  $V$  be a finite dimensional inner product space. An endomorphism

$f: V \rightarrow V$  is called self-adjoint if

$$\langle f(w), w \rangle = \langle v, f(w) \rangle \text{ for all } v, w \in V.$$

Example: Orthogonal projection

Let  $U \subset V$  be a subspace. As  $\langle, \rangle$  is positive definite,

$\langle, \rangle|_{U \times U}$  is nondegenerate. Thus we have

$V = U \oplus U^\perp$ . So every  $v \in V$  can be uniquely written as  $v = v_1 + v_2$  with  $v_1 \in U, v_2 \in U^\perp$

The orthogonal projection to  $U$  is  $p_U: V \rightarrow U, v = v_1 + v_2 \mapsto v_1$ .

This is clearly a linear map.

$p_U$  is self-adjoint: Let  $u_1, u_2 \in U, v_1, v_2 \in U^\perp$ , then

$$\langle p_U(u_1 + v_1), u_2 + v_2 \rangle = \langle u_1, u_2 + v_2 \rangle = \langle u_1, u_2 \rangle = \langle u_1 + v_1, u_2 \rangle = \langle u_1 + v_1, p_U(u_2 + v_2) \rangle.$$

Lemma: : Let  $V$  finite-dim inner product space.

$f: V \rightarrow V$  self-adjoint.

$$\text{Then } \text{im}(f) = \text{ker}(f)^\perp, \text{ker}(f) = \text{im}(f)^\perp$$

Proof: Set  $z = f(y) \in \text{im}(f)$ , let  $x \in \text{ker}(f)$ . Then

$$\langle x, f(y) \rangle = \langle f(x), y \rangle = \langle 0, y \rangle = 0. \text{ Thus } z \in (\text{ker } f)^\perp \text{ and } x \in \text{im}(f)^\perp$$

Thus  $\text{im}(f) \subset (\text{ker}(f))^\perp$ ,  $\text{ker}(f) \subset \text{im}(f)^\perp$ . By the

decomposition  $V = \text{im}(f) \oplus \text{im}(f)^\perp = \text{ker}(f) \oplus (\text{ker}(f))^\perp$  we get

$$\dim \text{im}(f)^\perp = \dim(\text{im } f) \geq n - \dim(\text{ker } f)^\perp \geq \dim \text{ker}(f). \text{ So they must be equal.}$$

In the same way  $\dim(\text{ker } f)^\perp = \dim \text{im}(f)$ . //

Now describe the matrices of self-adjoint maps with respect to an ONB.

Proposition. Let  $V$  be an inner product space. Let  $v_1, \dots, v_n$  be an ONB on  $V$ .

Then  $f$  is self-adjoint  $\Leftrightarrow$

with respect to this basis  $f$  is represented by a matrix  $A$  with  $A = \bar{A}^t$ , conjugate transpose. (if  $\mathbb{R} = \mathbb{R}$ )

Proof: If  $v = \sum x_i v_i$ ,  $w = \sum y_j v_j \in V$ , then  $f(v) = \sum z_i v_i$ , with

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \text{ Thus } \langle f(v), w \rangle = \langle \sum z_i v_i, \sum y_j v_j \rangle =$$

$$\sum_i z_i \bar{y}_i = \left( A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right)^t \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{pmatrix} = (x_1 \dots x_n) A^T \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{pmatrix}$$

$$\langle v, f(w) \rangle = (x_1 \dots x_n) \overline{A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}} = (x_1 \dots x_n) \bar{A} \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{pmatrix}.$$

Thus  $A^T = \bar{A}$ , and the argument can be reversed. //

Definition: A matrix  $A \in \text{Mat}(n, \mathbb{R})$  is called symmetric if  $A^t = A$

A matrix  $A \in \text{Mat}(n, \mathbb{C})$  is called

Hermitian if  $A^t = A$ , where  $A^* = \bar{A}^t$

So we see that w.r.t. an orthogonal basis

$f$  is self adjoint  $\Leftrightarrow A$  is symmetric ( $\mathbb{R} = \mathbb{R}$ )  
or Hermitian ( $\mathbb{R} = \mathbb{C}$ ).

Spectral theorem for self-adjoint operators.

We want to show that a self-adjoint operator on an inner product space has an orthonormal basis of eigenvectors, and all eigenvalues are real.

Lemma: Let  $\mathbb{R} = \mathbb{R}$  or  $\mathbb{C}$ . Let  $A \in \text{Mat}(n, \mathbb{R})$  satisfy  $A = \bar{A}^t$ . Let  $\lambda \in \mathbb{C}$  be a zero of  $p_A(x)$ .

Then  $\lambda \in \mathbb{R}$  and  $\lambda$  is an eigenvalue of  $A$ .

Proof: Denote  $\langle, \rangle$  the standard inner product on  $\mathbb{R}^n$ .

If  $\mathbb{R} = \mathbb{C}$ ,  $A$  is a Hermitian matrix on  $\mathbb{C}^n$ .

If  $\mathbb{R} = \mathbb{R}$ , then  $A$  is also a Hermitian matrix on  $\mathbb{C}^n$ .

Let  $\lambda$  be a zero of  $p_A(x)$ . Then  $\lambda$  is an eigenvalue of

$A: \mathbb{C}^n \rightarrow \mathbb{C}^n$ , Set  $v \in \mathbb{C}^n \setminus \{0\}$  with  $Av = \lambda v$ .

Then  $\lambda v = \langle Av, v \rangle = \langle v, Av \rangle = \bar{\lambda} v$ . Thus  $\lambda \in \mathbb{R}$ .

In case  $\mathbb{R} = \mathbb{C}$  we are done. In case  $\mathbb{R} = \mathbb{R}$ ,  $\lambda \in \mathbb{R}$  is still an eigenvalue of  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , as zero of  $p_A(x)$ .

Theorem: Set  $V$  be an inner product space of dim  $n$ .

Set  $f: V \rightarrow V$  self adjoint

Then there is an ONB of EV of  $V$   
and all the eigenvalues are real.

Proof: Induction on  $n = \dim V$ .

$n=0$  trivial

Note:  $f$  has a real eigenvalue: Choose an orthonormal basis of  $v_1, \dots, v_n$  of  $V$ , let  $A$  be the matrix of  $f$  w.r.t this basis, then  $A$  is Hermitian

Thus by the Lemma  $A$  has an eigenvector  $(x_1, \dots, x_n)$  with eigenvalue  $\lambda \in \mathbb{R}$ .

Thus  $u = \sum_{i=1}^n x_i v_i$  is an eigenvector of  $f$  with eigenvalue  $\lambda$ .

Set  $U := \langle u \rangle$ , as  $f$  is self-adjoint we have

$f(U^\perp) \subseteq U^\perp$ , because for  $v \in U^\perp$  we have

$$\langle f(v), u \rangle = \langle v, f(u) \rangle = \lambda \langle v, u \rangle = 0.$$

Then  $f|_{U^\perp}: U^\perp \rightarrow U^\perp, v \mapsto f(v)$  is a self-adjoint endomorphism of inner product space  $U^\perp$ .

And  $\dim U^\perp = n-1$ .

By induction  $U^\perp$  has an ONB  $v_2, \dots, v_n$  of EV

for  $f$  with real eigenvalues. Putting  $v_1 = \frac{u}{\|u\|}$

gives an ONB  $v_1, \dots, v_n$  of EV for  $f$  with real eigenvalues.  $\blacksquare$

Corollary: Set  $A \in \text{Mat}(n, \mathbb{R})$  be symmetric

(or  $A \in \text{Mat}(n, \mathbb{C})$  Hermitian).

Then there is a diagonal matrix  $D \in \text{Mat}(n, \mathbb{R})$

and an orthogonal matrix  $P \in \text{Mat}(n, \mathbb{R})$ , (or a unitary

matrix  $P \in \text{Mat}(n, \mathbb{C})$ ) with

$$A = P^* D P.$$

Proof: Choose an orthogonal basis  $(v_1, \dots, v_n)$  of  $\mathbb{R}^n$  consisting of eigenvalues to  $A$ . Set  $P = (p_{ij})$  be the change of basis matrix

$$v_i = \sum_{j=1}^n p_{ij} e_j$$

Then  $P$  is orthogonal and  $P^{-1} = P^*$ . and thus

$$A = P^* D P. \blacksquare \quad -31-$$

## Tensor products

We can "add" two vector spaces  $V, W$  by taking their direct sum  $V \oplus W$ .

If  $v_1, \dots, v_n$  is a basis of  $V$ ,  $w_1, \dots, w_m$  a basis of  $W$

then  $v_1, \dots, v_n, w_1, \dots, w_m$  is a basis of  $V \oplus W$ .

In particular  $\dim(V \oplus W) = \dim V + \dim W$ .

The tensor product  $V \otimes W$  allows to "multiply"

two vector spaces. Under the above assumptions

$(v_i \otimes w_j)_{i=1, j=1}^{n, m}$  is a basis of  $V \otimes W$ , in

particular  $\dim(V \otimes W) = n \cdot m$ .

The main purpose of the tensor product is to "linearize" multilinear maps.

To a bilinear map  $\psi: V \times W \rightarrow U$  corresponds

a linear map  $\tilde{\psi}: V \otimes W \rightarrow U$ .

(Recall bilinear means linear in both arguments.)

Definition: Set  $V, W$  be two vector spaces

A tensor product of  $V$  and  $W$  is a vector space  $T$  together with a bilinear map  $t: V \times W \rightarrow T$  with the following universal property:

For every vector space  $U$  and bilinear map  $\psi: V \times W \rightarrow U$ , there is a unique linear map  $\bar{\psi}: T \rightarrow U$  with  $\psi = \bar{\psi} \circ t$

$$\begin{array}{ccc} V \times W & \xrightarrow{\psi} & U \\ & \searrow t & \nearrow \bar{\psi} \\ & T & \end{array}$$

A tensor product, if it exists is unique up to unique isom.

Lemma: Set  $(T, t), (T', t')$  be two tensor products of  $V$  and  $W$

$\Rightarrow$  There exists a unique isom  $\iota: T \rightarrow T'$  s.t.  $t' = \iota \circ t$

$$\begin{array}{ccc} V \times W & \xrightarrow{t} & T \\ & \searrow t' & \downarrow \iota \\ & & T' \end{array}$$

Proof: Since  $t': V \times W \rightarrow T'$  is bilinear, by the univ. property of  $T$ , there is a unique linear map  $\iota: T \rightarrow T'$  making the diag. commute.

In the same way there is a unique linear map

$$c': T' \rightarrow T \text{ with } t = c' \circ t'.$$

Now both  $c' \circ c$  and  $\text{id}_T$  are linear maps satisfying

$$f \circ t = t, \text{ by univ. property } c' \circ c = \text{id}_T. \text{ In the same way}$$

$$c' \circ c = \text{id}_{T'} \quad \blacksquare$$

Because of this uniqueness we will in future talk about the tensor product  $V \otimes W$  of  $V$  and  $W$ .

Now we show the existence

Proposition: Let  $V, W$  be vector spaces.

Choose basis  $A$  of  $V$  and  $B$  of  $W$ .

Let  $T$  be the vector space with basis  $A \times B$ .

and define a bilinear map.

$$t: V \times W \rightarrow T \text{ via } t(a_i, b_j) = (a_i, b_j)$$

$$\text{i.e. } t\left(\sum \alpha_i a_i, \sum \beta_j b_j\right) = \sum_{i,j} \alpha_i \beta_j (a_i, b_j)$$

Then  $(T, t)$  is a tensor product of  $V$  and  $W$ .

Proof: Let  $\psi: V \times W \rightarrow U$  be bilinear.

To show there exists a unique linear map.

$$\bar{\Psi}: T \rightarrow U \quad \text{with} \quad \Psi = \bar{\Psi} \circ t.$$

Uniqueness: If  $\bar{\Psi}$  exists, then  $\bar{\Psi}(a, b) = \bar{\Psi}(t(a, b)) = \Psi(a, b)$  for  $a \in A, b \in B$ . This fixes the values of  $\bar{\Psi}$  on a basis of  $T$ , so  $\bar{\Psi}$  is unique.

Existence: Now define  $\bar{\Psi}(a, b) = \Psi(a, b)$  for all basis vectors. This defines a linear map  $\bar{\Psi}: T \rightarrow U$  and by definition  $\bar{\Psi} \circ t = \Psi$ . ~~///~~

In future write  $V \otimes W$  for the tensor product.

For elements  $v \in V, w \in W$  write  $v \otimes w$  for  $t(v, w)$ ,

thus  $t: V \times W \rightarrow V \otimes W, (v, w) \mapsto v \otimes w$  is the canonical bilinear map.

By definition if  $A$  is a basis of  $V, B$  a basis of  $W$ ,

then  $\{a \otimes b \mid a \in A, b \in B\}$  is a basis of  $V \otimes W$ .

Lemma: Let  $V, W$  be vector spaces. Let  $w_1, \dots, w_n$  be a basis of  $W$ . Then every element of  $V \otimes W$  can be written uniquely in the form:

$$v_1 \otimes w_1 + \dots + v_n \otimes w_n \quad \text{with} \quad v_1, \dots, v_n \in V.$$

Remark: Intuitively this means we can think of  $V \otimes W$  as "the vector space  $W$  with the scalars replaced by the elements of  $V$ ".

Proof: If  $A$  is a basis of  $V$ ,  $B$  a basis of  $W$ , then  $\{a \otimes b \mid a \in A, b \in B\}$  is a basis of  $V \otimes W$ .

In particular every element  $x \in V \otimes W$  can be written as  $x = y_1 \otimes z_1 + \dots + y_m \otimes z_m$  for  $y_1, \dots, y_m \in V, z_1, \dots, z_m \in W$

As  $w_1, \dots, w_n$  is a basis of  $W$ , we can write

$$z_j = \alpha_{j1} w_1 + \dots + \alpha_{jn} w_n \quad \alpha_{ji} \in \mathbb{R}.$$

By bilinearity of the map  $(y, z) \mapsto y \otimes z$  we get.

$$\begin{aligned} x &= y_1 \otimes (\alpha_{11} w_1 + \dots + \alpha_{1n} w_n) + \dots + y_m \otimes (\alpha_{m1} w_1 + \dots + \alpha_{mn} w_n) \\ &= (\alpha_{11} y_1 + \dots + \alpha_{m1} y_m) \otimes w_1 + \dots + (\alpha_{1n} y_1 + \dots + \alpha_{mn} y_m) \otimes w_n \end{aligned}$$

As required.

Uniqueness: Enough to show: if  $v_1 \otimes w_1 + \dots + v_n \otimes w_n = 0$

$$\text{Then } v_1 = \dots = v_n = 0.$$

Assume  $v_j \neq 0$ . Extend  $v_j$  to a basis  $u_1 = v_j, u_2, \dots$  of  $V$

Then  $v_1 \otimes w_1 + \dots + v_n \otimes w_n = v_j \otimes w_j + \text{linear combination of other basis vectors}$

Thus it is nonzero. //

$\oplus$  and  $\otimes$  behave a bit like addition and multiplication in a commutative ring.

Proposition: Set  $U, V, W$  be vector spaces.

We have isomorphisms.

$$U \otimes \mathbb{R} \cong U$$

$$U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$$

$$U \otimes V \cong V \otimes U$$

$$U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W)$$

Principle: To give a linear map  $f: U \otimes V \rightarrow W$  it is enough to define  $f(u \otimes v)$  for  $u \in U, v \in V$  o.l.s.

The map  $U \times V; (u, v) \mapsto f(u \otimes v)$  is bilinear.

Proof: By the universal property of  $U \otimes V$ , the bilinear maps  $g: U \times V \rightarrow W$  are in bijection to the linear maps  $f: U \otimes V \rightarrow W$ , via  $g(u, v) = f(u \otimes v)$ .  $\square$

Proof of proposition (mostly exercise)

$U \otimes V \cong V \otimes U$ : Define  $f: U \otimes V \rightarrow V \otimes U, u \otimes v \mapsto v \otimes u$ . (by principle). This is clearly an isom with  $f^{-1}(v \otimes u) = u \otimes v$ .

Leave the rest as exercise.  $\square$

We can form the tensor product of linear maps.

Definition: Set  $f: V \rightarrow W$ ,  $f': V' \rightarrow W'$  be linear maps.

Define  $f \otimes f': V \otimes V' \rightarrow W \otimes W'$  by  $f \otimes f'(v \otimes v') = f(v) \otimes f'(v')$

By the principle this defines a linear map

Remark: (1)  $\text{id}_V \otimes \text{id}_{V'} = \text{id}_{V \otimes V'}$

$$(2) (g \otimes g') \circ (f \otimes f') = (g \circ f) \otimes (g' \circ f')$$

If particular  $f, f'$  are both isom., then  $f \otimes f'$  is an isom. with inverse  $f^{-1} \otimes (f')^{-1}$ .

Proof: Exercise, check on elements  $v \otimes v'$ .

We can replace spaces of homomorphisms between vector spaces by tensor products.

Proposition: Set  $V, W$  be vector spaces.

There is a natural linear map

$$\phi: V^* \otimes W \rightarrow \text{Hom}(V, W), \quad \ell \otimes w \mapsto (v \mapsto \ell(v) \cdot w).$$

It is an isom when  $V$  or  $W$  is finite-dimensional

Proof:  $\phi$  is well defined and linear; By the principle

enough to see that  $V^* \times W \rightarrow \text{Hom}(V, W)$ ,  $(\ell, w) \mapsto (v \mapsto \ell(v) \cdot w)$  is bilinear. Linearity in  $w$  is clear

Linearity in  $v$  follows from vector space structure on  $V^*$ .

$$(\alpha_1 \ell_1 + \alpha_2 \ell_2, w) \mapsto (v \mapsto (\alpha_1 \ell_1 + \alpha_2 \ell_2)(v) \cdot w) = (\alpha_1 \ell_1(v) + \alpha_2 \ell_2(v)) \cdot w$$

linear in  $v$ .

We check bijectivity in case both  $V$  and  $W$  are finite dimensional.

Let  $v_1, \dots, v_m$  be a basis of  $V$  and  $v_1^*, \dots, v_m^*$  dual basis.

Define  $\phi': \text{Hom}(V, W) \rightarrow V^* \otimes W$ ,  $\phi'(f) = \sum_{i,j} a_{ij} v_i^* \otimes w_j$

where  $f(v_i) = \sum_j a_{ij} w_j \quad \forall i$

Exercise:  $\phi$  and  $\phi'$  are inverses to each other

Remark: Note that if  $v_1, \dots, v_m$  is a basis of  $V$   
 $w_1, \dots, w_n$  a basis of  $W$

Then  $\phi(\sum_{i,j} a_{ij} v_i^* \otimes w_j)$  is the linear map

$f: V \rightarrow W$  given in these basis by the matrix

$$A = (a_{ij})_{i=1, j=1}^{m, n}$$

In particular  $\text{id}_V = \phi(\sum_{i=1}^m v_i^* \otimes v_i)$ .

Lemma: Let  $V$  f.d. vector space. There is a linear form

$$T: V^* \otimes V \rightarrow \mathbb{R}, \quad T(\ell \otimes v) = \ell(v).$$

It makes the diagram

$$V^* \otimes V \xrightarrow{\phi} \text{Hom}(V, V) \quad \text{commute.}$$

$$T \searrow \mathbb{R} \swarrow \text{Tr} \quad , \text{ here Tr is the trace of } f$$

Proof: By the principle  $T$  is a well-defined linear map.

Let  $v_1, \dots, v_n$  be a basis of  $V$  with dual basis  $v_1^*, \dots, v_n^*$ .

If  $A = (a_{ij})_{i,j=1}^n$  is the matrix of  $f: V \rightarrow V$  w.r.t  $v_1, \dots, v_n$

$$\text{then } \text{Tr}(f) = \sum_{i=1}^n a_{ii}.$$

$$\text{On the other hand } \phi^{-1}(f) = \sum_{i,j=1}^n a_{ij} v_i^* \otimes v_j$$

By definition  $T(v_i^* \otimes v_j) = \delta_{ij}$ , thus

$$T \circ \phi^{-1}(f) = \sum_{i=1}^n a_{ii} = \text{Tr}(f).$$

## Symmetric and alternating products

We want to generalise the tensor product to deal with symmetric and multilinear maps.

Definition: Let  $V, W$  be vector spaces, let  $f: V^n \rightarrow W$  be a multilinear map.

(1)  $f$  is called symmetric if

$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = f(v_1, \dots, v_n) \text{ for all } v_1, \dots, v_n \in V \text{ and all } \sigma \in S_n.$$

(2)  $f$  is called alternating if  $f(v_1, \dots, v_n) = 0$  whenever  $v_i = v_j$  for some  $i \neq j$ .

Remark/Exercise: If  $\text{char } K \neq 2$ , then  $f$  is alternating if

$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \text{sign}(\sigma) f(v_1, \dots, v_n)$$

We want to define the analogue of the tensor product for symmetric and alternating multilinear forms, we do this by forcing (anti) commutativity on tensors.

Remark: While  $V^{\otimes n} = V \otimes \dots \otimes V$ , by associativity this is independent of bracketing. Easy induction shows:

(1) The map  $V^n \rightarrow V^{\otimes n}: (v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n$  is multilinear.

(2) A multilinear map  $\varphi: V^n \rightarrow W$  induces a unique linear map  $\bar{\varphi}: V^{\otimes n} \rightarrow W, v_1 \otimes \dots \otimes v_n \mapsto \varphi(v_1, \dots, v_n)$

Definition. Set  $W \subset V^{\otimes n}$  be the subspace spanned by all elements of the form.

$$v_1 \otimes v_2 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}, \text{ for}$$

$$v_1, \dots, v_n \in V, \sigma \in S_n$$

The quotient space  $W^{\otimes n} / W$  is called the  $n^{\text{th}}$  symmetric power of  $V$ . The image of  $v_1 \otimes \dots \otimes v_n$  in  $S(V)$  is denoted  $v_1 \cdots v_n$ . (Note by definition  $v_1 \cdots v_n = v_{\sigma(1)} \cdots v_{\sigma(n)}$  for  $\sigma \in S_n$ . We have forced the  $v_i$  to commute).

(2) Set  $W \subset V^{\otimes n}$  be the subspace spanned by all elements of the form  $v_1 \otimes \dots \otimes v_n$  for  $v_1, \dots, v_n \in V$  with  $v_i = v_j$  for some  $i \neq j$ .

The quotient space

$$\Lambda^n V = V^{\otimes n} / W \text{ is called the } n^{\text{th}} \text{ alternating power of } V.$$

The image of  $v_1 \otimes \dots \otimes v_n \in \Lambda^n V$  is denoted  $v_1 \wedge \dots \wedge v_n$

(again by definition  $v_1 \wedge \dots \wedge v_n = 0$  when  $v_i = v_j$  for some  $i \neq j$ .)

Theorem: (1) The map  $\varphi: V^n \rightarrow S^n V, (v_1, \dots, v_n) \mapsto v_1 \cdots v_n$  is multilinear and symmetric.

For every multilinear symmetric map  $f: V^n \rightarrow U$ .

There is a unique linear map

$$g: S^n(V) \rightarrow U, \text{ s.t. } f = g \circ \varphi, \text{ i.e.}$$

$$g(v_1 \cdots v_n) = f(v_1, \dots, v_n) \text{ for all } v_1, \dots, v_n \in V.$$

(2) Let  $\psi: V^n \rightarrow \wedge^n V, (v_1, \dots, v_n) \mapsto v_1 \wedge \dots \wedge v_n$  is multilinear and alternating

For every multilinear alternating map  $f: V^n \rightarrow U$  there is a unique linear map

$$g: \wedge^n V \rightarrow U \text{ with } f = g \circ \psi, \text{ i.e. } g(v_1 \wedge \dots \wedge v_n) = f(v_1, \dots, v_n)$$

Proof: We show (1), (2) is similar.

Clearly  $\varphi$  is multilinear, to check  $\varphi$  is symmetric

$$\text{we note } \varphi(v_{\sigma(1)}, \dots, v_{\sigma(n)}) - \varphi(v_1, \dots, v_n) = v_{\sigma(1)} \cdots v_{\sigma(n)} - v_1 \cdots v_n = 0.$$

Now let  $f: V^n \rightarrow U$  be multilinear and symmetric.

There is a unique linear map  $f': V^{\otimes n} \rightarrow U$  with  $f'(v_1 \otimes \dots \otimes v_n) = f(v_1, \dots, v_n)$ ,

By symmetry of  $f$  we have

$$f'(v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}) - f'(v_1 \otimes \dots \otimes v_n) = 0.$$

Thus  $f'|_W = 0$ . Therefore there is a unique linear map.

$$g: S^n(V) = V^{\otimes n} / W \rightarrow U, \quad g(v_1 \otimes \dots \otimes v_n) = f'(v_1 \otimes \dots \otimes v_n) = f(v_1, \dots, v_n)$$

Since for tensor products we can reformulate this as follows

Principle (1) To give a linear map  $f: S^k(V) \rightarrow W$

it enough to define  $f(v_1, \dots, v_k)$  for  $v_1, \dots, v_k \in V$  s.t.s.

The map  $V^k \rightarrow W; (v_1, \dots, v_k) \mapsto f(v_1, \dots, v_k)$  is multilinear and symmetric.

(2) To give a linear map  $f: \Lambda^k V \rightarrow W$

it enough to define  $f(v_1, \dots, v_k)$  for  $v_1, \dots, v_k \in V$  s.t.s.

The map  $V^k \rightarrow W; (v_1, \dots, v_k) \mapsto f(v_1, \dots, v_k)$  is multilinear and alternating.

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Linear maps  $f: V \rightarrow W$  induce maps between symmetric and alternating products.

Corollary (1) Set  $V, W$  vector spaces,  $f: V \rightarrow W$  linear.

There are linear maps

$$\Lambda^k f: \Lambda^k V \rightarrow \Lambda^k W, \quad \text{s.t. } (\Lambda^k f)(v_1 \wedge \dots \wedge v_k) = f(v_1) \wedge \dots \wedge f(v_k)$$

$$\text{Furthermore } \Lambda^k (g \circ f) = \Lambda^k g \circ \Lambda^k f, \quad \Lambda^k \text{id}_V = \text{id}_{\Lambda^k V} \\ (\Lambda^k f)^{-1} = \Lambda^k f^{-1} \quad -44-$$

(2) The corresponding statements hold for  $S^k V$ .

Proof: The map  $V^k \rightarrow \wedge^k W, (v_1, \dots, v_k) \mapsto f(v_1) \wedge \dots \wedge f(v_k)$  is multilinear and alternating. Thus we get a linear map.

$$\wedge^k f: \wedge^k V \rightarrow \wedge^k W, v_1 \wedge \dots \wedge v_k \mapsto f(v_1) \wedge \dots \wedge f(v_k).$$

The composition properties follow from uniqueness.

(Exercise) The proof of (2) is the same. //

For the rest of the lecture concentrate on alternating products.

We can multiply elements  $x$  in  $\wedge^k V$ ,  $y$  in  $\wedge^l V$  to  $x \wedge y$  in  $\wedge^{k+l} V$ .

So  $\bigoplus_{k \geq 0} \wedge^k(V)$  is some kind of noncommutative ring.

Proposition/Definition: There exists a bilinear map.

$$\wedge: \wedge^k V \times \wedge^l V \rightarrow \wedge^{k+l} V, (v_1 \wedge \dots \wedge v_k, v_{k+1} \wedge \dots \wedge v_{k+l}) \mapsto v_1 \wedge \dots \wedge v_{k+l}.$$

for all  $v_1, \dots, v_{k+l} \in V$ .

Denote this as  $\begin{matrix} (x, y) \\ \uparrow \quad \uparrow \\ \wedge^k V \quad \wedge^l V \end{matrix} \mapsto x \wedge y$ . It is called the wedge product of  $x$  and  $y$ .

Proof: Fix  $v_1, \dots, v_k \in V$ . Then the map

$\alpha_{v_1, \dots, v_k} : V^k \rightarrow \wedge^{k+l} V, (w_1, \dots, w_k) \mapsto v_1 \wedge \dots \wedge v_k \wedge w_1 \wedge \dots \wedge w_l$  is  
 $k$ -multilinear and alternating.

Thus there exists a unique linear map

$$\alpha_{v_1, \dots, v_k} : \wedge^k V \rightarrow \wedge^{k+l} V, w_1 \wedge \dots \wedge w_k \mapsto v_1 \wedge \dots \wedge v_k \wedge w_1 \wedge \dots \wedge w_l$$

Now define a map

$$\alpha : V^k \rightarrow \text{Hom}(\wedge^k V, \wedge^{k+l} V), \alpha(v_1, \dots, v_k) = \alpha_{v_1, \dots, v_k}$$

Again this is  $k$ -multilinear and alternating

Thus there is a unique linear map

$$\alpha : \wedge^k V \rightarrow \text{Hom}(\wedge^k V, \wedge^{k+l} V) \text{ with}$$

$$\alpha(v_1 \wedge \dots \wedge v_k) = \alpha_{v_1, \dots, v_k} \quad \text{Now define}$$

$$\alpha(x, y) = \alpha(x)(y) \quad \text{for } x \in \wedge^k V, y \in \wedge^l V$$

Finally we want to compute the dimension and a basis for  $\wedge^k V$ .

Proposition: Let  $V$  be a  $k$ -vector space of dim  $n$ .

Let  $B = (v_1, \dots, v_n)$  be a basis of  $V$ .

(1)  $\wedge^k V = \{0\}$  for  $k > n$ ,  $\dim \wedge^k V = \binom{n}{k}$  for  $0 \leq k \leq n$

(2) For  $0 \leq k \leq n$ , and  $I \subset \{1, \dots, n\}$  a subset of

cardinality  $k$ , write  $I = \{i_1, \dots, i_k\}$  with  $i_1 < \dots < i_k$ .

put  $v_I := v_{i_1} \wedge \dots \wedge v_{i_r}$ .

Then  $B_r = \{v_I \mid \text{card}(I) = r\}$  is a basis of  $\wedge^r V$ .

Proof: Clearly (2)  $\Rightarrow$  (1).

(2) The  $v_{i_1} \otimes \dots \otimes v_{i_r}$  with  $i_1, \dots, i_r \in \{1, \dots, n\}$  are a basis of  $V^{\otimes r}$ . By alternating property,

$v_{i_1} \wedge \dots \wedge v_{i_r} = 0$  if  $i_{j_1} = i_{j_2}$  for  $j_1 \neq j_2$ .

and  $v_{i_1} \wedge \dots \wedge v_{i_r} = \varepsilon(\sigma) v_{i_{\sigma(1)}} \wedge \dots \wedge v_{i_{\sigma(r)}}$ .

Thus the  $v_I$  with  $\text{card}(I) = r$  generate  $\wedge^r V$ .

To show: They are linearly independent.

For all  $I$  of  $\text{card}(I) = r$  let  $t_I \in \mathbb{R}$ . s.t.

$$\sum_I t_I v_I = 0 \quad I \text{ running through } r\text{-element subsets of } \{1, \dots, n\}.$$

Pick one such  $J$  and let  $K = \{1, \dots, n\} \setminus J$ . Complement.

Wedging by  $v_K$  gives

$$\sum_I t_I v_K \wedge v_I = 0.$$

If  $I \neq J$ , then  $I \cap K \neq \emptyset$ . Thus in  $v_K \wedge v_I$  one  $v_i$  is repeated, thus  $v_K \wedge v_I = 0$ .

Thus we get  $t_J v_K \wedge v_J = 0$ .

So it is enough to show that

$$v_x \wedge v_y \neq 0 \text{ in } \wedge^2 V.$$

This is a wedge product of 2 vectors in  $V$  which form a basis  $B$  of  $V$ . By the theorem on determinants there exists an  $n$  multilinear alternating map  $D: V^n \rightarrow \mathbb{R}$  s.t.  $D(B) = 1$ . Thus there exists a linear form  $D: \wedge^2 V \rightarrow \mathbb{R}$  with  $D(v_x \wedge v_y) = 1$ . Therefore  $v_x \wedge v_y \neq 0$ . Thus  $t_y = 0$  ~~///~~

Proposition: Let  $v_1, \dots, v_k$  be vectors in  $V$ . Then  $v_1, \dots, v_m$  are linearly independent  $\Leftrightarrow v_1 \wedge \dots \wedge v_k \neq 0$  in  $\wedge^k V$

Proof: " $\Rightarrow$ " Assume  $v_1, \dots, v_k$  are linearly independent.

Let  $v_{k+1}, \dots, v_n$  be vectors s.t.  $v_1, \dots, v_n$  are a basis of  $V$ .

Then  $v_1 \wedge \dots \wedge v_k$  is an element of a basis of  $\wedge^k V$ . Thus nonzero.

" $\Leftarrow$ " Assume  $v_1, \dots, v_k$  are linearly dependent. Find  $t_i \in \mathbb{R}$  not all 0

$$\sum_{i=1}^k t_i v_i = 0 \quad \text{can assume } t_j = 1.$$

$$\text{Then } v_j = -\sum_{i \neq j} t_i v_i$$

Then  $v_1 \wedge \dots \wedge v_k = -\sum_{i \neq j} v_1 \wedge \dots \wedge v_{j-1} \wedge v_i \wedge v_{j+1} \wedge \dots \wedge v_k = 0$ . Because  $v_i$  is repeated. ~~///~~