Elementary Algebraic Geometry Mini-Project

In all questions, the field k is algebraically closed.

1. Let k be a field and n a positive integer. Determine whether the set

$$\mathbf{GL}(n,k) := \{A \in \mathrm{Mat}(n \times n, k) \ : \ \det(A) \neq 0\}$$

   is an affine algebraic set.

2. Show that

   (a) the radical ideal $\sqrt{I}$ of a commutative ring R coincides with the intersection of all prime ideals of R containing I.

   (b) Use the ring $\mathbb{Z}$ to illustrate the equality in 2(a) above.

3. Let $A := k[x,y,z]$ be a polynomial ring over a field k and J an ideal of A such that

$$J := (x^2 + y^2 + z^2, xy + xz + yz).$$

   Identify $V(J)$ and $I(V(J))$.

4. Show that the variety $V(xy + z, x^2 - x + y^2 + yz) \subset k^3$ is reducible.

5. Determine the coordinate ring of the closed subvariety $\mathbf{SL}(2,k)$ of the set $\mathrm{Mat}(2 \times 2, k)$ of 2×2 matrices defined over a field k; i.e.,

$$\mathbf{SL}(n,k) := \{A \in \mathrm{Mat}(n \times n, k) \ : \ \det(A) = 1\}$$

6. Write the ideal $(x^3 - x, x^2 - y) \subset k[x,y]$ as the intersection of three prime ideals. Describe the corresponding geometry.