EAUMP-ICTP SCHOOL 2018: ADVANCED LINEAR ALGEBRA

The base field throughout is $\mathbb{C}$.

Project 1

Here V is an n-dimensional vector space, $A \in \mathrm{End}(V)$ and $M = \big(m_{ij}\big)_{i,j=1}^{n}$ is the corresponding matrix with respect to some choice of basis. Furthermore, we take $p_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ to be the characteristic polynomial of A.

(1) The trace of M is the sum of its diagonal entries denoted $\mathrm{Tr}(M)$ and defined by $\mathrm{Tr}(M) = \sum_{i=1}^{n} m_{ii}$. Show that the trace of M is invariant under conjugation by invertible matrices, i.e.

$$\mathrm{Tr}(M) = \mathrm{Tr}(P^{-1}MP).$$

Deduce that trace of A is well-defined.

(2) Show that the trace of A is given by $\mathrm{Tr}(A) = -a_{n-1}$ where a_{n-1} is the coefficient of x^{n-1} in the characteristic polynomial.

(3) Show that the determinant of A is given by $\det(A) = (-1)^n a_0$.

(4) (*Reality check*) Confirm that all the coefficients of the characteristic polynomial are invariant under conjugation and so well defined.

(5) Study the linear map

$$\bigwedge^n A\colon \bigwedge^n V \longrightarrow \bigwedge^n V.$$

Show that it coincides with multiplication by the determinant.

(6) So a_0 and a_{n-1} have some cool interpretation, it turns out that all the coefficients of the characteristic polynomial have cool interpretations

$$a_{n-i} = (-1)^i \,\mathrm{Tr}\Big(\bigwedge^i A\Big).$$

Is this relatively easy when A is diagonalisable? If so, can one try to lift the proof to all matrices?

(7) Now fix $n = 3$. Assuming the results above or otherwise, express $\mathrm{Tr}(A^4)$ in terms of the coefficients a_2, a_1, a_0 of the characteristic polynomial. Is it possible to do the same for $\mathrm{Tr}(A^5)$? Can this be generalised, i.e. for general n, can one express $\mathrm{Tr}(A^k)$ in terms of the coefficients $a_{n-1}, \ldots, a_0$?

Project 2

(1) Let $J_{y,n} \in \mathrm{Mat}(n, \mathbb{C})$ be the matrix with one eigenvalue y and a one Jordan block of size n, i.e. $J_{y,n} = (j_{ik})_{i,k=1}^n$ with $j_{ii} = y$, $j_{i(i+1)} = 1$ and all other entries 0. Prove that there are diagonal matrices arbitrarily close to $J_{y,n}$, i.e. for all $\epsilon > 0$ there is a diagonalisable matrix D such that $||J_{y,n} - M|| < \epsilon$.

Alternatively, you may want to do this in algebro-geometric terms, i.e. show that the subset of diagonalisable matrices is open in the affine variety of $n \times n$ matrices.

(2) Take a permutation $\sigma \in S_n$ and let P_σ be the $n \times n$-matrix given by permuting the columns of the identity matrix with σ, that is, if $P = (p_{ij})_{i,j=1}^n$ then $p_{ij} = 1$ if $j = \sigma(i)$ and 0 otherwise. Let D be a diagonal $n \times n$-matrix with diagonal entries $d_1, \ldots, d_n$, show that conjugation by P_σ gives a diagonal matrix $d_{\sigma(1)}, \ldots, d_{\sigma(n)}$.

(3) A partition of an integer n is a way of writing n as a sum of positive integers. For example, 4 has 5 partitions: (4); (3+1); (2+2); (2+1+1) and /(1+1+1+1). More formally we will define a partition to be a finite sequence of non-negative integers (λ_i) that is weakly decreasing

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k.$$

The size of a partition is the sum of its entries and is denoted $|\lambda|$.

The set of partitions of a fixed size n can be given a partial ordering as follows: for two partitions λ and μ of size n we say $\lambda \geq \mu$ if for all k

$$\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k.$$

Write down all the partitions of size 6 and the corresponding Hasse diagram (check Wikipedia for definition of Hasse diagram).

(4) Show that a matrix is nilpotent if and only if all its eigenvalues are 0.

(5) The group of invertible $n \times n$-matrices $\mathrm{GL}(n)$ acts on the set of nilpotent matrices by conjugation. Show that there is a natural bijection between the orbits of this action and partitions of size n. Given a partition λ we will use $\mathfrak{N}_\lambda$ to denoted the corresponding orbit.

(6) For a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ of n, let $N \in \mathfrak{N}_\lambda$. Show that

$$\mathrm{rank}(N^k) = \sum_{\lambda_i \geq k} (\lambda_i - k).$$

(7) (*Needs basic algebraic geometry from early next week*) Show that the set of matrices M with $\mathrm{rank}(M) \geq k$ forms a closed subvariety of the affine variety of $n \times n$-matrices. Also show that set of nilpotent matrices is a closed subvartiety of the affine variety of $n \times n$-matrices.

(8) The set of orbits of nilpotent matrices under conjugation is also a partially ordered set: given two orbits $\mathfrak{N}$ and $\mathfrak{M}$, we say $\mathfrak{N} \geq \mathfrak{M}$ if $\mathfrak{M}$ is contained in the closure of $\mathfrak{N}$. Use the above results to show that

$$\mathfrak{N}_\lambda \geq \mathfrak{N}_\mu \Longrightarrow \lambda \geq \mu.$$

(9) The other direction

$$\mathfrak{N}_\lambda \geq \mathfrak{N}_\mu \Longleftarrow \lambda \geq \mu$$

is true but a little harder to prove for me. Give it a good go! Come ask us if you need any hints.